

# On the Gelfand space of the measure algebra on the circle group

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November 26, 2014

## Abstract

This paper is devoted to studying certain topological properties of the maximal ideal space of the measure algebra on the circle group. In particular, we focus on Čech cohomologies of this space. Moreover, we show that the Gelfand space of  $M(\mathbb{T})$  is not separable. On the other hand, we give a direct procedure to recover many copies of  $\beta\mathbb{Z}$  in  $\mathfrak{M}(M(\mathbb{T}))$ , but we also show that this result is not accessible in the most natural way (namely, by the canonical mapping induced by homomorphism assigning to measure its Fourier - Stieltjes transform).

## 1 Introduction

Let  $M(\mathbb{T})$  denote the convolution algebra of complex, Borel and regular measures on the circle group. Also, let  $M_d(\mathbb{T})$  be the closed subalgebra of

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\*The research of this author has been supported by NCN grant no. N N201 607840

$M(\mathbb{T})$  consisting of discrete (purely atomic) measures while  $M_c(\mathbb{T})$  denotes the closed ideal of continuous (non-atomic) measures. It follows from the general Gelfand theory that  $\mathfrak{M}(M(\mathbb{T}))$  - the space of all maximal ideals or equivalently the space of all multiplicative - linear functionals is compact a Hausdorff space in the weak\* topology (those facts are proved in any textbook on Banach algebras - see for example [R2] or [Z]). We are going to give proofs of the basic topological properties of this space using analytical and algebraic methods. Let us recall some standard notations and basic facts which will be used in the sequel (for details see [Kat], [H] and [R1]). For  $\mu \in M(\mathbb{T})$  and  $n \in \mathbb{N}$  we define the  $n$ -th Fourier - Stieltjes coefficient of a measure  $\mu$  by the formula

$$\widehat{\mu}(n) = \int_0^{2\pi} e^{-int} d\mu(t).$$

It is well - known that the assignment  $\mu \mapsto \widehat{\mu}(n)$  is a multiplicative - linear functional for every  $n \in \mathbb{Z}$ . In the same manner, the mapping  $\mu \mapsto (\widehat{\mu})_{n=-\infty}^{\infty}$  is continuous homomorphism from  $M(\mathbb{T})$  to  $l^\infty(\mathbb{Z})$ . We will need the classical Wiener's lemma which connects Fourier - Stieltjes coefficients with atoms of the associated measure.

**Theorem 1** (Wiener's lemma). *Let  $\mu \in M(\mathbb{T})$ . Then*

$$\lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |\widehat{\mu}(n)|^2 = \sum_{\tau \in \mathbb{T}} |\mu(\{\tau\})|^2.$$

*In particular,  $\mu \in M_c(\mathbb{T})$  if and only if*

$$\lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |\widehat{\mu}(n)|^2 = 0.$$

The following proposition is an easy exercise from many textbooks on Banach algebras (see [Kan] and [R2])

**Proposition 2.** *Let  $A, B$  be complex, unital, commutative Banach algebras and  $f : A \mapsto B$  a continuous homomorphism. Then the mapping  $f^* : \mathfrak{M}(B) \mapsto \mathfrak{M}(A)$  defined by the formula*

$$f^*(\varphi)(a) = \varphi(f(a)) \text{ for } \varphi \in \mathfrak{M}(B)$$

*is continuous. Moreover, if the homomorphism  $f$  has dense range, then  $f^*$  is injective.*

We will also make use of notion of almost periodic sequence (for details and proofs of the next facts check [Kan])

**Definition 3.** We say that the sequence of complex numbers  $(a_n)_{n=-\infty}^{\infty}$  is almost periodic, if

$$\forall \varepsilon > 0 \exists p(\varepsilon) \in \mathbb{N} \forall I \exists m \in I \forall n \in \mathbb{Z} |a_{n+m} - a_n| < \varepsilon, \text{ where } I \text{ is any interval in } \mathbb{Z} \text{ of length } p(\varepsilon).$$

The set of all almost periodic sequences will be denoted by  $AP(\mathbb{Z})$ .

The most important properties of almost periodic sequences are summarized in the following theorem.

**Theorem 4.**  $AP(\mathbb{Z})$  is closed  $*$ -subalgebra of  $l^\infty(\mathbb{Z})$ .

A very useful link between measures and almost periodic sequences is given in the next proposition.

**Proposition 5.** The sequence of Fourier - Stieltjes coefficients of any discrete measure on  $\mathbb{T}$  is almost periodic.

## 2 Cech cohomologies

We begin this section with a simple proposition (in fact we work with 0 - dimensional Cech cohomologies - see [T]).

**Proposition 6.** There are only countably many open and closed subsets in  $\mathfrak{M}(M(\mathbb{T}))$ .

*Proof.* By the Shilov idempotent theorem for every open and closed subset  $U$  in  $\mathfrak{M}(M(\mathbb{T}))$  there exists an idempotent measure  $\mu \in M(\mathbb{T})$  such that  $\hat{\mu} = \chi_U$ . On the other hand, by Helson's theorem (see [H]) which characterises idempotents in  $M(\mathbb{T})$ , there are only countably many idempotents in  $M(\mathbb{T})$  (it follows from the fact that compact subgroups of  $\mathbb{T}$  are finite and there are only countably many of them).  $\square$

Despite the last proposition, we will show at the end of Section 3 that  $\mathfrak{M}(\mathbb{T})$  contains many copies of  $\beta\mathbb{Z}$  (which is extremally disconnected!) as closed topological subspaces.

**Proposition 7.** *There are a continuum of pairwise disjoint copies of  $\beta\mathbb{Z}$  in  $\mathfrak{M}(M(\mathbb{T}))$ .*

From general topology, if a compact space has a countable basis, then it has power at most continuum. But  $\beta\mathbb{Z}$  is a topological space of power  $2^c$  which based on two last propositions gives the following corollary.

**Corollary 8.**  *$\mathfrak{M}(M(\mathbb{T}))$  is not a totally disconnected topological space.*

This corollary can also be proved in different way - it follows from Shilov idempotent theorem that if the Gelfand space of commutative Banach algebra with unit is totally disconnected, then this Banach algebra is regular which is not the case for  $M(\mathbb{T})$ .

We move on to 1 - dimensional Čech cohomologies (a more detailed discussion of related topics is given in [T]).

**Definition 9.** *Let  $K$  be a compact Hausdorff space and let*

$$\text{Exp}(C(K)) = \{\exp(f) : f \in C(K)\}.$$

*Then  $\text{Exp}(C(K))$  is a closed subgroup of  $G(C(K))$  (group of invertible elements in  $C(K)$ ). We define 1 - dimensional Čech cohomologies  $H^1(K)$  as the quotient group*

$$H^1(K) := G(C(K))/\text{Exp}(C(K)).$$

*For a commutative, complex Banach algebra with unit  $A$  we define 1 - dimensional cohomologies of  $H^1(A)$  analogously:*

$$H^1(A) := G(A)/\text{Exp}(A).$$

It is an elementary fact that the Gelfand transform  $a \mapsto \hat{a}$  induces a homomorphism of  $H^1(A)$  into  $H^1(\mathfrak{M}(A))$ . However, the following theorem (not elementary at all!), with its proofs given in [A], [R] or [G] based on complex analysis in several variables, gives a much stronger statement.

**Theorem 10** (Arens-Royden). *If  $A$  is a complex, commutative Banach algebra with unit, then the map  $H^1(A) \mapsto H^1(\mathfrak{M}(A))$  induced by the Gelfand transform is an isomorphism.*

We will give two applications of the Arens-Royden theorem. The first one concerns  $\beta\mathbb{Z}$  (this fact is probably known in algebraic topology but using the above methods we will obtain it very quickly).

**Proposition 11.**  $H^1(\beta\mathbb{Z})$  is trivial.

*Proof.* Let us take  $(a_n)_{n=-\infty}^\infty \in G(l^\infty(\mathbb{Z}))$ . Remembering that

$$\sigma((a_n)_{n=-\infty}^\infty) = \overline{\{a_n : n \in \mathbb{Z}\}}$$

we conclude  $a_n \neq 0$  for every  $n \in \mathbb{Z}$ . Hence, there exists a sequence  $(b_n)_{n=-\infty}^\infty$  of complex numbers such that  $\exp(b_n) = a_n$  for every  $n \in \mathbb{Z}$ . We can also assume that the sequence of real numbers  $(\operatorname{Im} b_n)_{n=-\infty}^\infty$  is bounded. Now, we have  $|a_n| = |\exp(b_n)| = \exp(\operatorname{Re} b_n)$  for  $n \in \mathbb{Z}$ . The sequence  $(a_n)_{n=-\infty}^\infty$  is bounded and separated from 0 so we obtain  $c_1 \leq \operatorname{Re} b_n \leq c_2$  for some constants  $c_1, c_2 \in \mathbb{R}$  and every  $n \in \mathbb{Z}$  which proves  $(b_n)_{n=-\infty}^\infty \in l^\infty(\mathbb{Z})$ . Finally,

$$\begin{aligned} \exp((b_n)_{n=-\infty}^\infty) &= \sum_{k=0}^\infty \frac{(b_n^k)_{n=-\infty}^\infty}{k!} = \\ &= \left( \sum_{k=0}^\infty \frac{b_n^k}{k!} \right)_{n=-\infty}^\infty = (\exp(b_n))_{n=-\infty}^\infty = (a_n)_{n=-\infty}^\infty \end{aligned}$$

which shows that every invertible bounded sequence belongs to  $\operatorname{Exp}(l^\infty(\mathbb{Z}))$ . This finishes the proof with the aid of Arens-Royden theorem.  $\square$

The second application of Theorem 10 relates to  $\mathfrak{M}(M(\mathbb{T}))$ . It shows that this space is extremely complicated from the point of view of algebraic topology.

**Theorem 12.**  $H^1(M(\mathbb{T}))$  is uncountable.

*Proof.* Let us take  $\alpha \in \mathbb{T}$ ,  $\alpha \notin \pi\mathbb{Q}$ . Then  $\delta_\alpha \in G(M(\mathbb{T}))$ . We will show that the assumption  $\delta_\alpha \in \operatorname{Exp}(M(\mathbb{T}))$  leads to a contradiction. If  $\mu \in M(\mathbb{T})$  is such that  $\delta_\alpha = \exp(\mu)$ , then for every  $n \in \mathbb{Z}$  we have

$$\exp(-in\alpha) = \widehat{\delta}_\alpha(n) = \exp(\widehat{\mu}(n)).$$

Then,

$$-in\alpha = \widehat{\mu}(n) + 2\pi i l_n \text{ for some } l_n \in \mathbb{Z}.$$

Moreover, we easily see that we can assume that  $\mu$  is a discrete measure. Indeed, if  $\mu = \mu_c + \mu_d$  is the standard decomposition of a measure  $\mu$ , then

$$\delta_\alpha = \exp(\mu_c) * \exp(\mu_d) = (\exp(\mu_c) - \delta_0) * \exp(\mu_d) + \exp(\mu_d).$$

However  $\exp(\mu_c) - \delta_0$  is a continuous measure and the set of those measures is an ideal, which gives  $\exp(\mu_c) - \delta_0 = 0$  and hence  $\exp(\mu) = \exp(\mu_d)$ . From Proposition 5 we know that a sequence  $(\widehat{\mu}(n))_{n=-\infty}^{\infty}$  is almost periodic so there exists  $m \in \mathbb{N}_+$  such that for every  $n \in \mathbb{Z}$  the following inequality holds

$$|\widehat{\mu}(n+m) - \widehat{\mu}(n)| = |m\alpha + 2\pi(l_{n+m} - l_n)| < 1.$$

This gives:  $l_{n+m} - l_n = s$  for some  $s \in \mathbb{N}$  and every  $n \in \mathbb{Z}$ . Now, pick any  $n_0 \in \mathbb{Z}$  such that  $l_{n_0} \neq 0$ . Then for  $k \in \mathbb{N}$  we have

$$l_{n_0+mk} = l_{n_0+(m-1)k} + s = \dots = l_{n_0} + ks.$$

This leads to

$$|\widehat{\mu}(n_0 + mk)| = |k(m\alpha - 2\pi s) + n_0\alpha - 2\pi l_{n_0}|$$

and recalling  $\alpha \notin \pi\mathbb{Q}$  we obtain that a sequence  $(\widehat{\mu}(n_0 + mk))_{k=0}^{\infty}$  is unbounded which is the announced contradiction.

Finally, if  $\delta_\alpha$  and  $\delta_\beta$  belongs to the same coset of  $G(M(\mathbb{T}))$  with respect to  $\text{Exp}(M(\mathbb{T}))$ , then  $\delta_{\alpha-\beta} \in \text{Exp}(M(\mathbb{T}))$  and hence  $\alpha - \beta \in \pi\mathbb{Q}$ . This proves that  $H^1(M(\mathbb{T}))$  is uncountable since we cannot split  $\mathbb{T}$  into countably many countable parts.  $\square$

### 3 Main results

This section is devoted to proving two striking results concerning  $\mathfrak{M}(M(\mathbb{T}))$ .

#### 3.1 Fourier - Stieltjes sequences and ultrafilters

First one says that Fourier - Stieltjes sequences of measures from  $\mathbb{T}$  ‘glue’ some ultrafilters. The proof is based on the following proposition.

**Proposition 13.** *Let  $f : M(\mathbb{T}) \mapsto l^\infty(\mathbb{Z})$  be the homomorphism given by the formula  $f(\mu) = (\widehat{\mu})_{n=-\infty}^{\infty}$ . Then the dual mapping  $f^* : \mathfrak{M}(l^\infty(\mathbb{Z})) = \beta\mathbb{Z} \mapsto \mathfrak{M}(M(\mathbb{T}))$  is injective if and only if  $f$  has dense range.*

*Proof.* One implication (if  $f$  has dense range... ) is covered by Proposition 2 from the introduction.

Assume now that  $f^*$  is injective. Since we have an involution on  $M(\mathbb{T})$  given by  $\mu \mapsto \tilde{\mu}$  where  $\tilde{\mu}(E) = \overline{\mu(-E)}$  which has the property

$$\widehat{\tilde{\mu}}(n) = \overline{\widehat{\mu}(n)} \text{ for every } n \in \mathbb{Z}$$

we easily see that  $f(M(\mathbb{T}))$  is a  $*$  subalgebra of  $l^\infty(\mathbb{Z})$ . Recalling that  $l^\infty(\mathbb{Z})$  is  $*$  - isometrically isomorphic to  $C(\beta\mathbb{Z})$  (see for example [Kan]) we may treat  $f(M(\mathbb{T}))$  as a  $*$  - subalgebra of  $C(\beta\mathbb{Z})$ . By the Stone - Weierstrass theorem it is enough to verify that  $f(M(\mathbb{T}))$  separates points of  $\beta\mathbb{Z}$ , but this is exactly our assumption. Indeed, let us take two distinct ultrafilters  $\varphi_1, \varphi_2 \in \beta\mathbb{Z}$ . Then, by the assumption  $f^*(\varphi_1)(\mu_1) \neq f^*(\varphi_2)(\mu_2)$  for some  $\mu_1, \mu_2 \in M(\mathbb{T})$ . Equivalently, we have

$$\varphi_1((\mu_1)_{n=-\infty}^\infty) \neq \varphi_2((\mu_2)_{n=-\infty}^\infty)$$

which is the desired assertion by the definition of the Gelfand transform of an element in Banach algebra.  $\square$

Now, we are in position to prove the main theorem.

**Theorem 14.** *Let  $f : M(\mathbb{T}) \mapsto l^\infty(\mathbb{Z})$  be a homomorphism as in the previous proposition. Then  $f^*$  is not injective.*

*Proof.* By the Proposition 13 it is enough to prove that  $f(M(\mathbb{T}))$  is not dense in  $l^\infty(\mathbb{Z})$ . This fact was established in [DR] using the notion of weakly almost periodic sequence, but our proof is much more elementary. Let us take the sequence  $(a_n)_{n=-\infty}^\infty$  defined as follows:  $a_n = 1$  for  $n \geq 0$  and  $a_n = 0$  for  $n < 0$ . It is easily seen that the sequence  $(a_n)_{n=-\infty}^\infty$  is not almost periodic. Hence, by Theorem 4 there exists  $c > 0$  such that

$$\inf_{\mu \in M(\mathbb{T})} \sup_{n \in \mathbb{Z}} |\widehat{\mu}_d(n) - a_n| \geq \inf_{(b_n) \in AP(\mathbb{Z})} \sup_{n \in \mathbb{Z}} |a_n - b_n| > c \quad (1)$$

Let us assume the contrary, that  $f(M(\mathbb{T}))$  is dense in  $l^\infty(\mathbb{Z})$  and fix  $\delta > 0$ . Then there exists  $\mu \in M(\mathbb{T})$  such that

$$\sup_{n \in \mathbb{Z}} |\widehat{\mu}(n) - a_n| < \delta.$$

Splitting  $\mu$  into its discrete  $\mu_d$  and continuous part  $\mu_c$  we have for every  $n \in \mathbb{Z}$

$$\delta > |\widehat{\mu}(n) - a_n| = |\widehat{\mu}_c(n) + \widehat{\mu}_d(n) - a_n| \geq ||\widehat{\mu}_c(n)| - |\widehat{\mu}_d(n) - a_n||,$$

which gives for every  $n \in \mathbb{Z}$

$$|\widehat{\mu}_c(n)| \geq |\widehat{\mu}_d(n) - a_n| - \delta. \quad (2)$$

Now, by the Inequality (1), without losing generality, there exists  $n_0 \in \mathbb{N}_+$  (if  $n_0 \in \mathbb{Z}_-$  the same argument works) such that

$$|\widehat{\mu}_d(n_0) - a_{n_0}| > \frac{c}{2}.$$

Now, we use the fact that the sequence  $(\widehat{\mu}_d(n))_{n=-\infty}^{\infty}$  is almost periodic (Proposition 5). Hence, for fixed  $\varepsilon > 0$  we can find in every interval in  $\mathbb{Z}$  of the form  $(kp(\varepsilon), (k+1)p(\varepsilon)]$  an integer  $m_k$  such that  $|\widehat{\mu}_d(n_0+m_k) - \widehat{\mu}_d(n_0)| < \varepsilon$  for every  $k \in \mathbb{N}$ . Recalling that  $a_n = 1$  for  $n \geq 0$  we obtain for every  $k \in \mathbb{N}$

$$\begin{aligned} |\widehat{\mu}_d(n_0 + m_k) - a_{n_0+m_k}| &= |\widehat{\mu}_d(n_0 + m_k) - a_{n_0}| = \\ &= |\widehat{\mu}_d(n_0 + m_k) - \widehat{\mu}_d(n_0) + \widehat{\mu}_d(n_0) - a_{n_0}| \geq \\ &= |\widehat{\mu}_d(n_0) - a_{n_0}| - |\widehat{\mu}_d(n_0 + m_k) - \widehat{\mu}_d(n_0)| > \frac{c}{2} - \varepsilon. \end{aligned}$$

This estimation together with (2) gives for every  $k \in \mathbb{N}$

$$|\widehat{\mu}_c(n_0 + m_k)| > \frac{c}{2} - \varepsilon - \delta.$$

On the other hand, from Wiener's lemma (by passing to a subsequence) we get

$$\lim_{k \rightarrow \infty} \frac{1}{2kp(\varepsilon) + 1} \sum_{n=-kp(\varepsilon)+n_0}^{kp(\varepsilon)+n_0} |\widehat{\mu}_c(n)|^2 = 0$$

But in every interval  $[-kp(\varepsilon) - n_0, kp(\varepsilon) + n_0]$  there are  $k$  integers for which the summed expression is greater than  $(\frac{c}{2} - \varepsilon - \delta)^2$ . This gives

$$\begin{aligned} &\frac{1}{2kp(\varepsilon) + 2n_0 + 1} \sum_{n=-kp(\varepsilon)-n_0}^{kp(\varepsilon)+n_0} |\widehat{\mu}_c(n)|^2 \geq \\ &\frac{1}{2kp(\varepsilon) + 2n_0 + 1} k \left( \frac{c}{2} - \varepsilon - \delta \right)^2 \geq \frac{(\frac{c}{2} - \varepsilon - \delta)^2}{3p(\varepsilon) + 2n_0}, \end{aligned}$$

which is the desired contradiction.  $\square$



## 3.2 Non-separability

Now, we move on to the second main theorem of this paper, namely that the maximal ideal space of the measure algebra on the circle group is not separable.

We shall need the following simple arithmetic lemma which follows easily from the fact that every element in a lacunary sequence with ratio at least 3 is greater than twice of sum of all previous elements in the sequence.

**Lemma 15.** *Let  $(n_k)_{k=1}^\infty$  be sequence of positive integers such that  $\frac{n_{k+1}}{n_k} \geq 3$  for every  $k \in \mathbb{N}$  and for any set  $A \subset \{n_k : k \in \mathbb{N}\}$ , let us write  $\tilde{A}$  for the set defined as follows*

$$\tilde{A} = \left\{ \sum_{l=1}^n \varepsilon_l a_l : \varepsilon_l \in \{-1, 0, 1\}, a_l \in A, n \in \mathbb{N} \right\}.$$

*With these notions, if  $A \cap B$  is finite, then  $\tilde{A} \cap \tilde{B}$  is finite.*

We will also make use of a well-known observation due to Sierpiński.

**Proposition 16.** *There exists uncountably many infinite subsets of positive integers such that intersection of each two is finite.*

Now, we recall a few facts on Riesz products, that is continuous probabilistic measure on the circle of group of the following form

$$R(a_k, n_k) = \prod_{k=1}^{\infty} (1 + a_k \cos(n_k t)),$$

where this infinite product is meant as weak\* limit of finite products. From the construction of Riesz products we have (for simplicity we write  $\mu = R(a_k, n_k)$  and  $A = \{n_k : k \in \mathbb{N}\}$ )

$$S(\mu) := \{n \in \mathbb{Z} : \hat{\mu}(n) \neq 0\} = \tilde{A}.$$

For a sequence of natural numbers  $(n_k)_{k=1}^\infty$  we assume  $\frac{n_{k+1}}{n_k} \geq 3$  for every  $k \in \mathbb{N}$  and from  $(a_k)_{k=1}^\infty$  we demand  $-1 < a_k \leq 1$  for  $k \in \mathbb{N}$ . We will use the following strong result on Riesz products (for a proof consult [BM] or [GM]).

**Theorem 17** (Brown,Moran). *If  $(a_k)_{k=1}^\infty$  is a sequence of real numbers satisfying  $-1 < a_k \leq 1$  for  $k \in \mathbb{N}$  with the property*

$$\forall_{n \in \mathbb{N}} \sum_{k=1}^{\infty} |a_k|^n = \infty$$

*then the Riesz product  $R(a_k, n_k)$  has all convolution powers mutually singular.*

It is an elementary result from the general theory of Banach algebras (see for example [GM]), that for Riesz products satisfying the assumptions of Theorem 17 we have

$$\{z \in \mathbb{C} : |z| = 1\} \subset \sigma(R(a_k, n_k)).$$

In fact, a much stronger result is true (see [BBM]).

**Theorem 18** (Brown,Bailey,Moran). *If  $\mu \in M(\mathbb{T})$  is a hermitian measure with all convolution powers mutually singular then*

$$\sigma(\mu) = \{z \in \mathbb{C} : |z| \leq r(\mu)\}.$$

The theorem of Zafran will also be appropriate (check [Z]).

**Theorem 19** (Zafran). *Let  $\mathcal{C} = \{\mu \in M_0(\mathbb{T}) : \sigma(\mu) = \widehat{\mu}(\mathbb{Z}) \cup \{0\}\}$ .*

1. *If  $\varphi \in \mathfrak{M}(M_0(\mathbb{T})) \setminus \mathbb{Z}$ , then  $\varphi(\mu) = 0$  for all  $\mu \in \mathcal{C}$ .*
2.  *$\mathcal{C}$  is closed ideal in  $M_0(\mathbb{T})$ .*
3.  *$\mathfrak{M}(\mathcal{C}) = \mathbb{Z}$ .*

We are ready now to prove the announced theorem.

**Theorem 20.**  *$\mathfrak{M}(M(\mathbb{T}))$  contains uncountably many open disjoint open subsets. In particular,  $\mathfrak{M}(M(\mathbb{T}))$  is not separable.*

*Proof.* Let us take any sequence  $(n_k)_{k=1}^\infty$  of positive numbers such that  $\frac{n_{k+1}}{n_k} \geq 3$  for  $k \in \mathbb{N}$ . Then by Proposition 16 there exists uncountably many subsets  $\{A_\alpha\}_{\alpha \in \mathbb{R}}$  of  $A = \{n_k : k \in \mathbb{N}\}$  with the property  $A_{\alpha_1} \cap A_{\alpha_2}$  is finite for any  $\alpha_1, \alpha_2 \in \mathbb{R}$ ,  $\alpha_1 \neq \alpha_2$ . For  $\alpha \in \mathbb{R}$  we define the sequence  $(a_k^\alpha)_{k=1}^\infty$  by the

formula:  $a_k^\alpha = 1$  for  $k \in A_\alpha$  and  $a_k^\alpha = 0$  otherwise. We assign to every  $A_\alpha$  the Riesz product

$$\mu_\alpha := R(a_k^\alpha, n_k) = \prod_{k=1}^{\infty} (1 + a_k^\alpha \cos(n_k t)).$$

Then  $S(\mu_\alpha) = \widetilde{A}_\alpha$ . By the assumption and Lemma 15 we obtain  $S(\mu_{\alpha_1}) \cap S(\mu_{\alpha_2})$  is finite for  $\alpha_1, \alpha_2 \in \mathbb{R}$ ,  $\alpha_1 \neq \alpha_2$ . Now,

$$S(\mu_{\alpha_1} * \mu_{\alpha_2}) = S(\mu_{\alpha_1}) \cap S(\mu_{\alpha_2}) \text{ is finite .}$$

Hence  $\mu_{\alpha_1} * \mu_{\alpha_2}$  is a trigonometric polynomial and it is easy to prove that all trigonometric polynomials belong to  $\mathcal{C}$ . Let us define for  $\mu \in M(\mathbb{T})$

$$\widetilde{S}(\mu) = \{\varphi \in \mathfrak{M}(M(\mathbb{T})) : \widehat{\mu}(\varphi) \neq 0\}.$$

We also recall that  $\mathfrak{M}(M(\mathbb{T})) = \mathfrak{M}(M(\mathbb{T})) \setminus h(M_0(\mathbb{T})) \cup \mathfrak{M}(M_0(\mathbb{T}))$  where

$$h(M_0(\mathbb{T})) = \{\varphi \in \mathfrak{M}(M(\mathbb{T})) : \varphi(\mu) = 0 \text{ for all } \mu \in M_0(\mathbb{T})\}.$$

Let us fix  $\alpha_1, \alpha_2 \in \mathbb{R}$ ,  $\alpha_1 \neq \alpha_2$  and take  $\varphi \in \mathfrak{M}(M(\mathbb{T})) \setminus \mathbb{Z}$ . If  $\varphi \in \widetilde{S}(\mu_{\alpha_1})$ , then  $\varphi(\mu_{\alpha_1} * \mu_{\alpha_2}) = 0$  which follows from the theorem of Zafran if  $\varphi \in \mathfrak{M}(M_0(\mathbb{T}))$  and just from the definition, if  $\varphi \in \mathfrak{M}(M(\mathbb{T})) \setminus h(M_0(\mathbb{T}))$ . From this we get  $\varphi \notin \widetilde{S}(\mu_{\alpha_2})$  which we can summarize as

$$\widetilde{S}(\mu_{\alpha_1}) \cap \widetilde{S}(\mu_{\alpha_2}) \cap \mathfrak{M}(M(\mathbb{T})) \setminus \mathbb{Z} = \emptyset \text{ for } \alpha_1 \neq \alpha_2. \quad (3)$$

Using Theorem 18 we know that there exists  $z \in \mathbb{C} \setminus \mathbb{R}$  in  $\sigma(\mu_\alpha)$  and, recalling that  $\widehat{\mu_\alpha}(\mathbb{Z}) \subset \mathbb{R}$ , we are able to find an open neighborhood  $U$  of  $z$  which does not intersect the real line. Since  $\widehat{\mu_\alpha} : \mathfrak{M}(M(\mathbb{T})) \mapsto \mathbb{C}$  is a continuous function we get  $\widehat{\mu_\alpha}^{-1}(U) = T_\alpha$  is an open set contained in  $\mathfrak{M}(M(\mathbb{T})) \setminus \mathbb{Z}$ . On the other hand,  $T_\alpha \subset \widetilde{S}(\mu_\alpha)$  and hence  $T_{\alpha_1} \cap T_{\alpha_2} = \emptyset$  for  $\alpha_1 \neq \alpha_2$  by (3) which finishes the proof.  $\square$

In the same manner we prove the following corollary which explains why determining spectra of measures is so difficult task.

**Corollary 21.** *There exist no countable set of multiplicative linear functionals on  $M(\mathbb{T})$  such that the spectrum of any measure from  $M(\mathbb{T})$  is a closure of the values of its Gelfand transform restricted to this set.*

We are ready now to give a proof of Proposition 7 (we follow notation from the last proof).

Let  $U, V \in \beta A_\alpha$  be two different ultrafilters. Then there exists  $X \in U$  and  $Y \in V$  such that  $X \cap Y = \emptyset$ . Let  $\mu_X$  and  $\mu_Y$  be the Riesz products built on the sets  $X$  and  $Y$  respectively. Obviously we have  $\hat{\mu}_X(n) = 1/2$  for  $n \in X$  and, by Lemma 15,  $\hat{\mu}_Y(n) = 0$  for sufficiently big  $n \in X$ . Therefore  $\lim_U \hat{\mu}_X(n) = 1/2$  while  $\lim_V \hat{\mu}_X(n) = 0$ . Hence the map  $\Lambda : \beta A_\alpha \rightarrow \mathfrak{M}(M(\mathbb{T}))$  given by the formula  $\Lambda_\alpha(U)(\mu) = \lim_U \hat{\mu}(n)$  is injective. Then  $\Lambda_\alpha(\beta A_\alpha)$  is a homeomorphic copy of  $\beta\mathbb{N}$ . Exactly in the same way we prove that  $\Lambda_\alpha(A_\alpha) \cap \Lambda_\beta(A_\beta) = \emptyset$  for  $\alpha \neq \beta$ .

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